

# Tools for Decision Making under Uncertainty

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**Abstract.** Decision making is a process used in many parts of life to determine an optimal choice with respect to a particular subjective aim for a particular decision maker. In this paper we focus on two often considered distinct aims, namely maximizing of an utility function (e.g. an investment profit) and getting a more reliable global description of considered situation based on observed data (e.g. the final outcome of databases merging). In both cases we face the problem, that the data are unreliable, since they contain uncertainty caused by their source (i.e. human being). If we are looking for the optimum of the former aim, a game theory reformulation of the decision making task brings a smoother way to reach it. If the latter aim is considered, a merging procedure (also called fusion) processing the data should help us. This aim was chosen as the author's present work is connected with it and she has to inspect the state of the art. This paper describes four recently developed methods dealing with decision making under uncertainty in two considered directions and one tool used for comparison of the fusion algorithms.

## Introduction

In common life almost every minute we have to make a decision satisfying our subjective aims. The procedure leading to the best decision among the possible ones is called decision making. Here we focus on two distinct aims, which one may have almost surely considered in the past. The first one is the maximization of an utility function (e.g. investment profit), the second one is to get a more reliable and correct description of considered environment (e.g. about the occurrence of considered events). Since in both cases the observations are required, handling of data sources (e.g. human beings, sensors) is important part of decision making. Every source has its limiting abilities, e.g. precision of the measuring sensor or human ability to provide complete information, therefore given data contain a part reflecting the source's uncertainty. The seminal work of Wald on decision making under uncertainty dates back to 50's (see Wald [1950]), since then people try to deal with this issue by developing different methods.

In the case when maximal profit is of interest a game theory formulation of original decision making task should properly treat the uncertainty and give us a smoother and more consistent way leading to a solution (see Neumann and Morgenstern [1944]). The uncertainty here is caused by a non-cooperative relation among the sources, which means they have no chance to observe the choices of other sources. In Reneke [2009] author suggests a two player game, DM (a decision maker) and incompletely observable and unpredictable source called NATURE. The uncertainty is treated with functions from the sigmoid family. To find the optimal choice (i.e. investment alternative), authors introduce a decision variable based on second order statistics of the score for particular investment alternative. In Madani and Lund [2011] a higher number of non-cooperative sources is considered. But none of them is of type NATURE from the previous case. Here the uncertainty is handled by interpretation of the matrix (subject to criteria and alternatives) of sources' payoff via ordinal rank matrix and by Monte Carlo simulations. To obtain the optimal game solution, the ordinal rank matrix is transformed into game theory terms (via transition matrix) and several stability definitions are applied.

If we are interested in a global description of considered events (hypotheses) and we have corresponding observations, their fusion leads to a satisfactory solution. The uncertainty comes through the impreciseness of the data given by sources. In Fassinut-Mombot and Choquel [2004] a method using basic terms of information theory is introduced, e.g. entropy and mutual information. Authors suggest to reduce the space of given observations for a particular hypothesis via introducing the notion of source's redundancy (equivalently via source's complementarity). To do that a probabilistic representation of the observations obtained via maximum entropy principle (Shore and Johnson [1980]) is used. The optimal hypothesis is determined by conditional entropy. Note that here the uncertainty is measured via Shannon's entropy (see Shannon [1948]). In Pavlin et al. [2010] the hypotheses are represented via random variables and the conditional probabilities, then modeled by Bayesian networks and described by factor graphs. Authors introduce a processing unit operating on the subset of all variables, which satisfies

a cooperative scenario within the sources operating at least one common variable. Then the updated versions of (marginal) posterior probabilities of random variables are computed. Finally, we take a look at an information-based quality measure (see Qu et al. [2002]), which serves for comparison of the fusion algorithms by computing a mutual information between the input observations and output. Although this concept is simple and seems to work well, under specific conditions it gives unsatisfactory results, as shown in Chen et al. [2008].

In the following sections we give a brief review of these methods together with their subjective critique.

## Game theory reformulation

### Reformulation under uncertainty and risk

In this section we take a look on the formulation proposed in Reneke [2009]. Here, the decision maker's attention is paid to long term investments connected with increasing oil prices and environmental degradation (i.e. degradation on air). The aim is to construct a sequence of rational investment decisions, which keeps a balance between decision maker's payoff and risk (its gain and loss).

Since the oil prices and air degradation are almost unpredictable in long time period, the construction of such a sequence is a difficult problem. To solve this author suggests a two-player game between the decision maker and NATURE. The strategies of these players are:

- DECISION MAKER (DM) – rational investment decisions,
- NATURE – future oil prices and environmental (air) degradation.

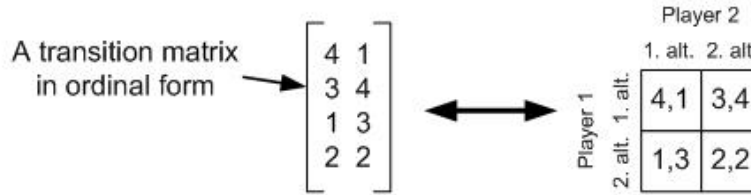
The uncertainty here comes through the fact, that at the time DM makes the investment decision, NATURE's strategy is unknown to it. Following the Knight's distinctions (see Knight [1921]), the uncertainty is considered as an unquantifiable variable (there is no assumption on existence of describing distribution), while the risk is estimated in terms of a quantifiable variable (with a distribution). Their proposed procedure is the following:

- construct an outcome (denoted by  $Z$ ) of a random performance of one of DM's investment alternatives; this outcome depends on time and NATURE's strategies (via the investment alternative),
- construct the score of the game (denoted by  $V$ ); this depends on the outcome from the previous step and on a discount rate  $r$ ,
- for later purposes: compute the expectation and the variance of  $V$  (both are still depending on NATURE's strategies),
- make an assumption on the form of NATURE's strategies – each is described by single parameter sigmoid function; the original uncertainties are now reduced to uncertainties in the pace of change – the timing (determined by a particular parameter) of NATURE's strategy performance is unknown (there are two different parameters, because two original uncertainties are modeled separately).

The computations now depend on the choice of timing parameters; in the paper authors considered three values for each parameter and computed parts of decision variable (introduced later) for all possible pairs of parameters. Since the uncertainties have already been modeled, the main focus is now on the construction of decision variable.

Since the outcome  $Z$  is normally distributed with mean  $\mu$  and standard deviation  $\sigma$  dependent on DM's alternative, the bad outcome is introduced as  $\{Z \leq \mu - \alpha\sigma\}$ , where  $\alpha$  is some positive number determined by DM's risk tolerance. The previously mentioned investment risk is then interpreted as the probability of a bad outcome and is taken as a constant. Then for each DM's alternative we can compute the corresponding  $\mu_i$  and  $\sigma_i$ . If we have to decide between two alternatives  $i$  and  $j$  and the criterion  $\mu_i - \sigma_i > \mu_j - \sigma_j$  holds for all considered possibilities of timing parameters, we should take the alternative  $i$ . The only arising problem is that the inequality between alternatives can change at another time point, so the choice of the best alternative is ambiguous. The authors suggest two criteria, both following the relation between alternatives: one on the whole set of possibilities for timing parameters, the other based on particular values of timing parameter. If the arising set of possible investment alternatives has more than one element you will have to use the least squares method to find the best one.

Here only the case of two uncertainties is treated. Multiple uncertainty situation can also be addressed, since every larger problem can be decomposed into (already proposed) smaller problems.



**Figure 1.** Relationship between multi-criteria decision making form and game theory form.

Conclusion: Through the paper many strict assumptions were made: e.g. the uncertainty is modeled by a specific one-parameter function (to simplify the computation of the variables based on it), where also the parameter values were set. Finally, we get a deterministic model, describing a specific situation, which obviously can not be used generally.

### A Monte Carlo game theory approach

In this section we take a look on another formulation of decision making task under uncertainty (see Madani and Lund [2011]). In contrary to the previous case, the proposed method:

- treats multiple sources, multiple criteria and obviously multiple alternatives,
- considers only sources with the ability to provide the information about possible alternatives; none of them is unpredictable as the source NATURE (see previous section).

Again, the sources are non-cooperative, which means none of them can see the choices of the other sources through the game. The non-cooperative scenario was preferred to a perfect cooperation because the latter allows agreement only on one alternative and a cooperative outcome – any kind of disagreement leading to a non-cooperative outcome is disregarded. As we will see later, the proposed method considers both types of outcomes.

If we assume the situation with two DMs, one criterion and two alternatives, then the main idea of this game-theory reformulation is:

1. Construct a performance (or utility) matrix  $P$  for all considered alternatives under particular criterion of a particular source (here we get  $2 \times (1 \times 2) = 2 \times 2$  matrix). This is a conventional form of multi-criteria decision making; for the considered situation there are only two outcomes, which occur when both sources agree on the same alternative.
2. Construct a new  $2 \times 2$  matrix corresponding to  $P$ , where the ordinal ranks are used instead of utilities – the elements are ranks of a particular alternative with respect to the criterion of a concrete DM.

3. Construct a transition matrix to convert the proposed problem into a game theoretic form. A transition matrix is now a matrix with the number of rows corresponding to the number of all possible combinations (strategies) of DMs' alternatives and with the number of columns corresponding to the number of players. Each row describes a possible outcome, each column represents a player. The whole matrix represents the payoff of two players from four possible outcomes – the elements of the matrix are represented by ordinal ranks (e.g. a higher rank coincides with higher payoff). Thus we have now a  $4 \times 2$  matrix (see Fig. 1 on the left). The transition matrix corresponds to a  $2 \times 2$  game, shown in Fig. 1 on the right.

Now we want to find the possible results of the game. This step has the following pattern:

- use several stability definitions (Madani and Hipel [2011]) to determine, which possible outcome is the most likely to occur,
- find the outcome determined by majority of stability conditions – this one has a higher chance to be a final outcome of the game.

So far we did not stress that the final results based on ordinal representation of information (see step 2.) are less sensitive to uncertainty provided by DMs. It is because the results are insensitive to the changes in the performance as long as the rankings do not change. By transformation into game theory terms there is no need to weight DMs and criteria (to determine which one is more or less useful), which also reduces the influence of uncertainty on results. The last step to eliminate the remaining uncertainty consists of Monte Carlo simulations (multiple computations), where authors:

- choose a random set of performances,
- randomly select utilities of the four available strategies for each considered DM,
- construct the transition matrix,
- solve the game with 6 non-cooperative stability definitions.

Conclusion: This approach brings the simplification in the representation of original situation by introducing the ordinal rank matrix followed by the game theory reformulation. In contrast with the previous method, it has just one special assumption – a non-cooperative scenario between the sources, which in fact generalizes a multi-criteria decision making. The only problem brings the last part, the elimination of uncertainty via Monte Carlo simulations, which is computationally very time-demanding and is to be improved.

## Information fusion

Information fusion (or merging) is another way of how the decision making faces uncertainty. It is a process of combining information from heterogeneous sources in order to get more reliable information describing the whole considered environment (e.g. events or hypotheses). The main issue of the information fusion is how to treat the incompleteness, impreciseness and uncertainty contained in processed information. In this section we briefly describe three methods introducing different kinds of fusion processes.

But first, let us recall some of the notions of information theory:

- entropy  $H(X)$  measures uncertainty contained in the random variable  $X$ ; usually it refers to Shannon's entropy (see Shannon [1948]) and evaluates the expected value of information contained,
- maximum entropy principle MEP (see Shore and Johnson [1980]) states, that from all of the distributions describing the considered environment and satisfying particular constraints the one with the highest entropy should be chosen,
- conditional entropy  $H(Y|X)$  measures the amount of information provided by the output  $Y$  when the input  $X$  of the fusion system is known,
- mutual information  $I(X, Y)$  measures the statistical dependence between two random variables  $X$ ,  $Y$  and the amount of information that one variable contains about the others.

## Entropy fusion model

Fassinut-Mombot and Choquel [2004] introduce a method using all terms mentioned above to determine the best information description of the considered environment. The main idea is to reduce the combination space (space of all inputs  $X$ ) by introducing a notion of source's redundancy and complementarity. In particular, authors use mutual information and conditional entropy to make a decision, which information sources to merge. They follow the decomposition of the entropy of an output  $Y$  as follows:

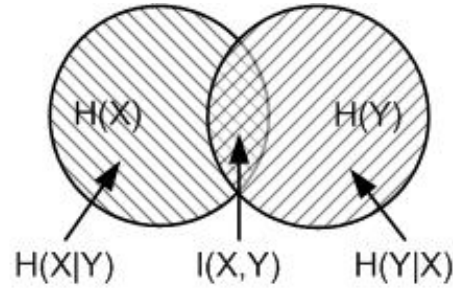
$$I(X, Y) + H(Y|X) = H(Y) = \text{constant} \quad (\text{see Fig. 2}),$$

where  $I(X, Y)$  allows us to measure the redundancy of transmitted information and  $H(Y|X)$  allows us to measure the complementarity of the information. In order to optimize the fusion system:

- maximize the mutual information  $I(X, Y)$  between all inputs  $X$  and the output vector  $Y$  (which coincides with minimization of the redundancy of information sources),
- or equivalently minimize the conditional entropy  $H(X|Y)$  (which coincides with maximization of the complementarity between output and inputs).

The main steps of the method are:

- Modeling step:
  - construct the set including all elementary events relevant to the given problem (which stands for the set of all possible hypotheses),



**Figure 2.** Relationship among simple entropy, conditional entropy and mutual information.

- represent the knowledge via conditional probability of hypothesis' characteristic conditioned on observed data; to choose the optimal one use MEP (note, that by using MEP also the uncertainty is respected),
- Combination step: use conditional entropy of each hypothesis conditioned on the corresponding set of observations for a reduction of the combination space – choose the observations for which the conditional entropy is minimal,
- Decision making step: choose the hypothesis with the lowest conditional entropy (conditioned on the corresponding reduced set of observations); if ambiguity occurs, measure a quality of the decision based on mutual information between the corresponding reduced combination space and hypothesis to determine the optimal hypothesis.

Conclusion: This work brings interesting procedure for fusion of information sources. However, there is a concern about the applicability. In the beginning a very strong assumption is made: the set of all possible hypotheses is given. If the hypotheses are not specifically given by somebody interested in the result of the merging, their formulation might be a very difficult task.

### A mutli-agent systems approach

The problem proposed in the previous section should be overcome by the method proposed in Pavlin et al. [2010]. When using real-world describing sources for decision making, it is impossible to determine the set of possible events. Therefore it is suggested that the critical hidden events must be expressed via fusion of an enormous amount of uncertain and heterogeneous information (e.g. obtained from sensors). The technique consists of the following (simplified) steps:

- assume that the relevant states (realizations of events) can be described through a finite number of random variables,
- model the hypotheses (its realizations cover hidden and observed states) by conditional probability conditioned on observations; use the interpretation via Bayesian networks to compute the probabilities,
- interpret the current situation via probabilistic graphical models: the evidence of instantiated variables should bring the simplification in computation of the joint probability over the set of all hypotheses,
- introduce a fusion agent – a processing unit operating on the variables in its Bayesian network (a subset of all considered hypotheses) and cooperating with other agents,
- update the probability of a particular random variable by computing the posterior probability based on information from sources containing this variable in their set of hypotheses.

Conclusion: Since this method treats a very general situation, it is obvious that its structure is more complicated. One can complain, that under the amount of allowed uncertainty in this case, it is hard to obtain, e.g., the probabilities based on often unreliable data. Despite that, under several assumptions like conditional independence, this method brings useful result.

### Information based quality measure and its theoretical analysis

Because many methods of information fusion are being developed, one can not be sure which one to use in order to get the best output. Therefore, there arises the need to measure the quality of the output based on particular input. One of such quality measures was introduced in Qu et al. [2002]. It is based on mutual information between the output  $F$  and inputs  $A, B$  from the sources respectively as follows:

$$Q = I(f, a) + I(f, b)$$

and it is shown on an example of image fusion of two image sources, where  $a, b$  denote corresponding pixels in the image given by the first and the second source, respectively;  $f$  stands for the fused pixel. The symbol  $I(f, a)$  (similarly  $I(f, b)$ ) refers to mutual information:

$$I(f, a) = \sum_{f,a} p(f, a) \log \frac{p(f, a)}{p(f)p(a)},$$

where  $p(f, a)$  stands for a joint distribution of random variables  $A, F$  and  $p(f)p(a)$  is the distribution associated with the independence of  $A$  and  $F$ .

Several fusion schemes were compared by using this quality measure in Qu et al. [2002]. In a recent paper (Chen et al. [2008]), the weighted averaging of two images was analyzed. For the analytical studies the intensity of each pixel is introduced and is modeled it by Gaussian random vector and Gaussian noise. The study shows that under image degradation the quality measure  $Q$  gives antipodal results to what is usually expected. In particular, if we add more noise to the source images, the quality measure  $Q$  increases (although we would expect the opposite).

Conclusion: Be careful when comparing different fusion techniques via quality measures. The main issue is to make sure we compare the methods at the same image level, at different levels it has different meaning (see Qu et al. [2002]). Also pay attention to the results the quality measures are giving, since occurrence of an unexpected behavior for a specific situation is highly possible.

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